found that a slightly modified conjugate gradient method using a numerical approximation of the partial derivatives worked relatively well, and took about 6 min (on the average) to converge. In any case, we did succeed in finding the optimum two-burn solution for a range of final initial orbits—final orbits with periods of 12 and 24 h and perigee altitudes of 200–900 n.mi. and initial orbits with altitude 150 n.mi. and inclinations of 45 and 57 deg. Table 2 shows the optimal solutions we calculated for the various values of P and r_p for 57 deg initial inclination—the total ΔV(broken into ΔV_1, ΔV_2), the size of the transfer orbit (a_T/a_f), the values of the two independent variables $(\Delta\Omega, \theta_T)$, and the firing angles (γ, ψ). For an initial inclination of 45 deg, the ΔVs_{total} (for the range of perigee attitudes) are of the order of 15,600 ft/s for a 12-h orbit and around 1000 ft/s higher for a 24-h orbit.

The results show, first of all, that the two-burn method always requires a significantly smaller ΔV than the simplified three-burn (as was shown in Table 1). It is also obvious that the minimum total ΔV depends strongly on the initial inclination. The dependence on the final perigee altitude, on the other hand, is very weak—a difference of 100 n.mi. changes the total ΔV by only a few feet per second.

Conclusions

It is clear that within the parameters studied, the two-burn method is considerably more efficient than the simplified, step-by-step three-burn method. While the ΔV can be reduced further by making a small plane change at the first (perigee) burn or by using an optimized three-burn solution,[1] our two-burn approximation (which is easily implemented on a small computer) is accurate enough for preliminary design purposes, and essentially optimum for plane changes less than 10 deg.

Acknowledgment

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Optimal Proportional Navigation

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Introduction

UNTIL now, the investigation of optimal guidance laws has been restricted to the purely proportional type,[1,2] wherein the commanded accelerations are applied normal to the missile velocity and both missile and target are assumed to have constant speeds. Because of the inherent differences in the equations of motion, these results cannot be applied normal to the line of sight (LOS). Under true proportional navigation (TPN), the commanded missile acceleration applied normal to the LOS, a_{M_c}, is equal to $u\dot{\theta}$, where u is a proportionality constant and $\dot{\theta}$ the LOS rate. Using the exact nonlinear equations of motion,[3,4] it has been shown that the time of capture is reduced as u increases; however, the control expenditure also increases at the same time. A reasonable compromise seems to be possible if a quadratic performance index is used to optimize a weighted combination of the time of capture and the expenditure of maneuvering energy. Based on the exact nonlinear equations of motion in the plane, an optimal trajectory of u for a vehicle pursuing a maneuvering target is derived in this paper. It is assumed that a complete knowledge of the target's motion is available to the missile.

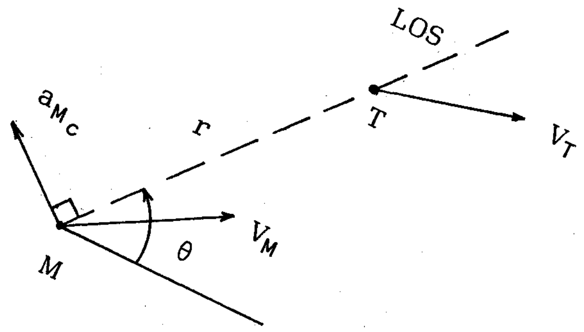

Fig. 1 Planar pursuit geometry.

Problem Formulation

A missile M is attempting to capture a target T as shown in Fig. 1. The vehicles have velocity components V_{M_r}, $V_{M\theta}$, V_{T_r}, and $V_{T\theta}$, respectively, in the polar coordinate system (r, θ) with the origin fixed at the missile. The equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = \dot{V}_{T_r} \qquad r(0) = r_0 \tag{1a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \dot{V}_{T\theta} - u\dot{\theta} \qquad \theta(0) = \theta_0 \tag{1b}$$

where r is the range of the target and θ the aspect angle measured with respect to an inertial reference. Equivalently, we have

$$\dot{V}_r = V_\theta\dot{\theta} + \dot{V}_{T_r} \qquad V_r(0) = V_{r_0} \tag{2a}$$

$$\dot{V}_\theta = -V_r\dot{\theta} + \dot{V}_{T\theta} - u\dot{\theta} \qquad V_\theta(0) = V_{\theta 0} \tag{2b}$$

where $V_r = \dot{r} = V_{T_r} - V_{M_r}$ and $V_\theta = r\dot{\theta} = V_{T\theta} - V_{M\theta}$ are the components of the target velocity relative to the missile. By changing the independent variable from t to θ, Eqs. (2) are reduced to the simple form

$$V_r' = V_\theta + V_{T_r}' \qquad V_r(\theta_0) = V_{r_0} \tag{3a}$$

$$V_\theta' = -V_r + V_{T\theta}' - u \qquad V_\theta(\theta_0) = V_{\theta_0} \tag{3b}$$

where V_{T_r}' and $V_{T\theta}'$ are given continuous functions of θ and primes denote differentiation with respect to θ.

It has been shown[5] that θ is a monotonically increasing or decreasing function of t, depending on the sign of V_{θ_0} (we will assume a positive V_{θ_0} in the following discussion). Thus, the problem to be solved is to find a control u such that the capture point $r(\theta_f) = 0$ is assured while the performance index

$$J = \theta_f + \rho\int_{\theta_0}^{\theta_f} u^2 \, \mathrm{d}\theta$$

is minimized. The weighting factor ρ is selected in such a way that the required acceleration is realistic. The range of the parameter ρ and the initial conditions (V_{r_0}, V_{θ_0}) that ensure the

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capture-point condition are characterized in the following derivation.

Optimal Control Solution

The optimization problem is solved by applying the maximum principle.[6] Define an additional state variable ξ such that

$$\xi' = 1 + \rho u^2 \tag{4}$$

with $\xi(\theta_0) = \theta_0$. Then, the Hamiltonian for the system of Eqs. (3) and (4) becomes

$$H = \lambda_{V_r}(V_\theta + V'_{T_r}) + \lambda_{V_\theta}(-V_r - u + V'_{T_\theta}) + \lambda\xi(1 + \rho u^2) \tag{5}$$

The adjoint variables are defined by the equations

$$\lambda'_{V_r} = \lambda_{V_\theta} \tag{6a}$$

$$\lambda'_{V_\theta} = -\lambda_{V_r} \tag{6b}$$

$$\lambda'_\xi = 0 \tag{6c}$$

with the boundary conditions given by

$$\lambda_{V_r}(\theta_f) = 0 \tag{7a}$$

$$\lambda_{V_\theta}(\theta_f) = \text{free} \tag{7b}$$

$$\lambda_\xi(\theta_f) = -1 \tag{7c}$$

The adjoint system, Eqs. (6), can be solved immediately as

$$\lambda_{V_\theta} = A\cos(\theta - \theta_f) \tag{8a}$$

$$\lambda_{V_r} = A\sin(\theta - \theta_f) \tag{8b}$$

The optimal control law $\partial H/\partial u = 0$ gives

$$u^* = \frac{\lambda_{V_\theta}}{2\lambda_\xi \rho} = -\frac{A}{2\rho}\cos(\theta - \theta_f) \tag{9}$$

and the resulting missile acceleration is

$$a_{Mc} = -\frac{A}{2\rho}\dot{\theta}\cos(\theta - \theta_f)$$

Instead of a constant, as in the case of true proportional navigation, the optimal control u^* is now a cosine function of the aspect angle θ; however, in the neighborhood of the pursuit end, we have $\cos(\theta - \theta_f) \doteqdot 1$ and the optimal control u^* can be approximated by a constant. This, in turn, implies that the optimality of the conventional proportional navigation is only valid in the vicinity of the pursuit end.

Substituting the optimal control u^* into Eqs. (3), we have the following expressions for the state variables V_r and V_θ:

$$V_r(\theta) = B\cos(\theta - \theta_f) + C\sin(\theta - \theta_f) + \frac{A}{4\rho}(\theta - \theta_f)\sin(\theta - \theta_f) + \int_{\theta_0}^{\theta}\sin(\theta - \phi)f(\phi)\,d\phi \tag{10a}$$

$$V_\theta(\theta) = -V'_{T_r} + C\cos(\theta - \theta_f) + \left(\frac{A}{4\rho} - B\right)\sin(\theta - \theta_f) + \frac{A}{4\rho}(\theta - \theta_f)\cos(\theta - \theta_f) + \int_{\theta_0}^{\theta}\cos(\theta - \phi)f(\phi)\,d\phi \tag{10b}$$

where $f(\theta) = V'_{T_\theta}(\theta) + V''_{T_r}(\theta)$ and the four parameters A, B, C, and θ_f can be determined in terms of V_{r_0}, V_{θ_0}, and ρ by the initial conditions

$$V_r(\theta_0) = V_{r_0}, \qquad V_\theta(\theta_0) = V_{\theta_0} \tag{11}$$

and the terminal conditions

$$H(\theta_f) = 0, \qquad V_\theta(\theta_f) = 0 \tag{12}$$

Note that $V_\theta = r\dot{\theta}$ and the condition $V_\theta(\theta_f) = 0$ guarantees that the capture point is reached with a finite LOS rate, i.e., with a finite expenditure of maneuvering energy. To ensure a finite time of capture, one additional condition is needed; namely,

$$V_r(\theta_f) < 0$$

This inequality characterizes the capture area for V_{r_0}, V_{θ_0}, and ρ that is compatible with the capture-point condition.

To obtain the time history of θ, we define the specific angular momentum

$$h = r^2\dot{\theta} \tag{13}$$

By introducing this new variable in Eq. (1b) and using the optimal control u^*, we have

$$\dot{h} = V_\theta(-u^* + V'_{T_\theta}) = h(-u^* + V'_{T_\theta})/r$$

or

$$dh/h = (-u^* + V'_{T_\theta})V_\theta^{-1}\,d\theta$$

After integration, this yields

$$h = h_0\,e^{-\Phi(\theta)} \tag{14}$$

where

$$\Phi(\theta) = \int_{\theta_0}^{\theta} -\frac{V'_{T_\theta}(\phi) - u^*(\phi)}{V_\theta(\phi)}\,d\phi$$

With this equation and the relation $\dot{\theta} = V_\theta^2/h$, we have

$$t = \int_{\theta_0}^{\theta}\frac{h_0 e^{-\Phi(\phi)}}{V_\theta^2(\phi)}\,d\phi \tag{15}$$

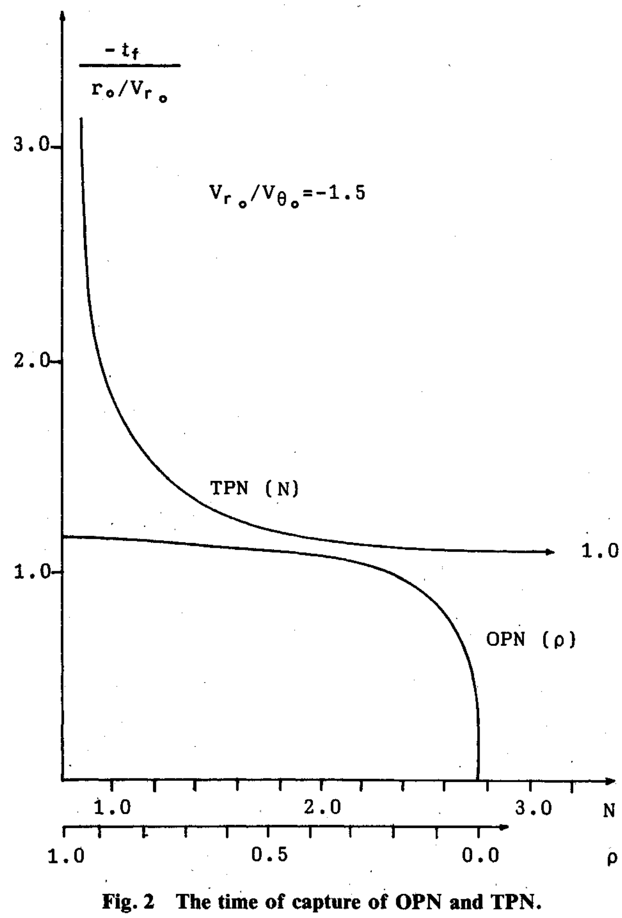

Fig. 2 The time of capture of OPN and TPN.

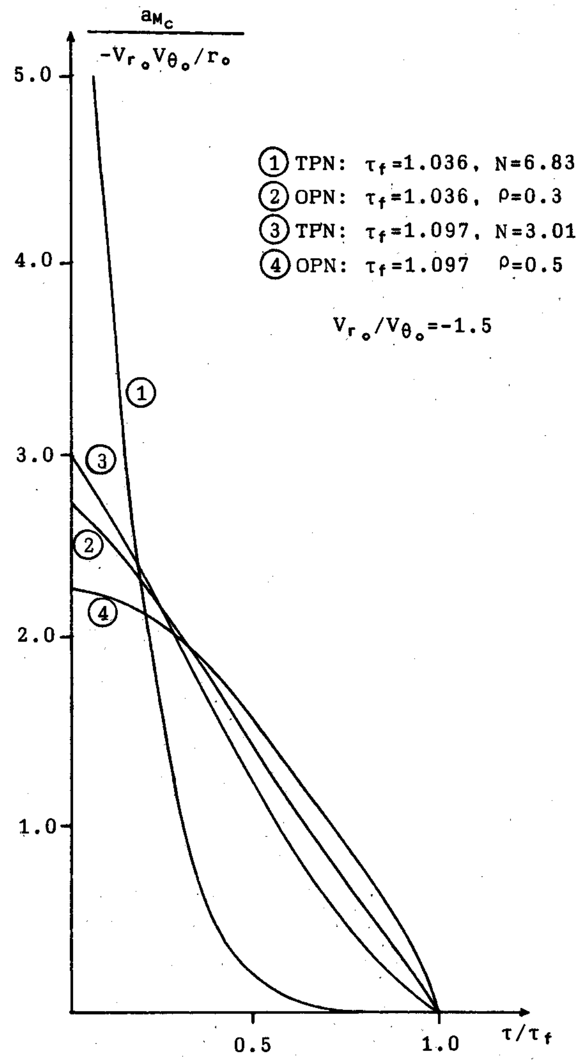

Fig. 3 The time histories of commanded missile acceleration for OPN and TPN.

As mentioned previously, t is a monotonically increasing function of θ and vice versa, provided $\dot{\theta}_0$ is positive. The time of capture is obtained by letting $\theta = \theta_f$ in Eq. (15).

Numerical Results

Simulation has been employed to demonstrate the superiority of optimal proportional navigation (OPN). A comparison was made with TPN in which the commanded missile acceleration is given by

$$a_{M_c} = -NV_{r_0}\dot{\theta} \tag{16}$$

and N is the navigation constant.

For comparison, a nonmaneuvering target was considered. The time of capture is depicted in Fig. 2, where the curve for TPN is obtained by the formula[4]

$$\tau_f = \frac{-t_f}{r_0/V_{r_0}} = 1 - \frac{1}{1 + (V_{r_0}/V_{\theta_0})^2\,(1-2N)} \tag{17}$$

and the corresponding τ_f for OPN can be found by the use of Eq. 15. It is seen that over most of the range of ρ and N, which are compatible with the capture-point condition, the time of capture of OPN is lower than that of TPN. Especially, when $(V_{r_0}/V_{\theta_0})^2\,(1-2N)$ is close to -1, TPN requires a time that is almost several times as long as that required by OPG. The lowest posible τ_f that can be attained by TPN is equal to 1, but the τ_f can be reduced arbitrarily near 0 by OPG.

To compare the acceleration properties, we keep the time of capture the same in both the laws by the special choices of ρ and N, and the resulting time histories of commanded missile accelerations are depicted in Fig. 3. It can be observed that maximum required acceleration for OPN control is reduced largely from the TPN requirement.

Conclusion

The problem of finding a nonlinear optimal guidance law for a homing missile with commanded acceleration applied normal to the LOS so as to capture a maneuvering target with a predetermined trajectory, while minimizing a weighted linear combination of time of capture and energy expenditure, has been solved in closed form. The derived optimal control law is equal to the LOS rate multiplied by a cosine function of aspect angle, which can be implemented easily. From the numerical simulation, the resulting guidance law appears to yield a significant advantage over true proportional navigation.

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Dynamics and Control of a Space Platform with a Tethered Subsatellite

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Introduction

THE analyses of the dynamics and control of the tethered subsatellite system (TSS) have been performed by a host of investigators. It was noted that for local vertical stationkeeping, within the linear range, tether tension would not provide control of the out-of-orbit-plane swing motion, but such control would be implemented in the nonlinear system due to higher-order coupling, or by including nonlinear feedback terms in the tension-control law. Bainum and Kumar[1]

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